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On the exponentials of some structured matrices

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Abstract

This paper provides explicit techniques to compute the exponentials of a variety of structured 4×4 matrices. The procedures are fully algorithmic and can be used to find the desired exponentials in closed form. With one exception, they require no spectral information about the matrix being exponentiated. They rely on a mixture of Lie theory and one particular Clifford algebra isomorphism. These can be extended, in some cases, to higher dimensions when combined with techniques such as Givens rotations.

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1. Introduction

Finding matrix exponentials is arguably one of the most important goals of mathematical physics. In full generality, this is a thankless task [1]. However, for matrices with structure, finding exponentials ought to be more tractable. In this paper, confirmation of this phenomenon is given for a large class of 4×4 matrices with structure. These include skew-Hamiltonian, perskewsymmetric, bisymmetric (i.e., simultaneously symmetric and persymmetric, e.g., symmetric, Toeplitz), symmetric and Hamiltonian etc. Some of the techniques presented extend almost verbatim to some families of complex matrices (see remark 3.4, for instance).

There are at least three important sources which motivate the problem of exponentiating a matrix in applications. First, the problem of solving a linear system of differential equations clearly requires matrix exponentiation. Such linear systems may represent the actual dynamics of the system or may be obtained from Jacobian linearization of a nonlinear system. They could also arise as part of specific ansätze to solve other systems. Second, many applications call for explicit parametrizations of classes of matrices. One such parametrization is afforded by the exponentials of related classes of structured matrices. A third source is the study of numerical structure preserving algorithms for nonlinear differential equations which evolve on Lie groups [2]. Many such algorithms have, as an essential ingredient, the computation of certain matrix exponentials. Hence, explicit formulae for matrix exponentials are of vital utility to all such problems. Bearing in mind these motivations, a brief list of applications featuring the structured matrices, being studied in this work, is provided next. Orthogonal (and, hence skew-symmetric) matrices are ubiquitous in many applications. Novel examples include robotics, computer vision, various types of lattice filters such as thin film dielectric coatings and Fabry-Perot etalon structures [3–5]. Hamiltonian matrices are central to classical and quantum mechanics, of course. In particular, the metaplectic representations of their exponentials intervene in an essential way in quantum optics [6]. Symmetric, persymmetric matrices arise in vibrational analysis, the study of neural networks [7, 8]. Toeplitz matrices arise in signal processing and a variety of inverse problems. Finally, many of these matrices (or rather faithful representations of them) play a role in the study of (quasi) exactly solvable models in quantum mechanics. In particular, the exponentials of such matrices intervene in various disentangling formulae. This list is, of course, minuscule and does not even touch upon novel fields of applications such as biology.

In light of the above, it is interesting that the exponentials of these structured matrices can be calculated algorithmically in dimension four (these lead to closed form formulae), for the most part, without any auxiliary information about their spectrum. For general symmetric matrices, however, the spectral decomposition of a 3×3 matrix is needed (see, however, (iii) of remark 2.1). On the other hand, this spectral decomposition can itself be produced in closed form. Thus, even for such matrices the techniques described here can be justifiably called closed form methods. For brevity, this paper only records explicit algorithms for finding these matrix exponentials—the resultant final formulae can easily be written once the reported procedures are implemented.

The methods discussed below are of two types. The first, which is more versatile, relies on an algebra isomorphism of real 4×4 matrices with $H \otimes H$, where H is the field of quaternions. This algebra isomorphism, known from the theory of Clifford algebras and which ought to be widely advertized, was used in a series of interesting articles by Mackey *et al* [9–11] for finding eigenvalues of some of the structured matrices discussed here. The present paper can be seen as a contribution of a similar type. It is emphasized that for the preponderance of the matrices, considered here, *this algebra isomorphism alone is needed*. In particular, in this paper no use is made of any of the structure preserving rotations used in [9–11] ever—see (ii) of remark 3.2. The second is based on the observation that several 'covering' space Lie group homomorphisms, when made explicit, contain in them a recipe for finding exponentials of matrices belonging to certain Lie algebras. This circumstance renders the exponentiating of some 4×4 matrices (real/complex) equivalent to the job of finding the exponential of 2×2 matrices—which can be done in closed form. This method is, however, applicable only to a limited family of matrices. Therefore, this method is presented in an appendix.

It is worth noting that, though most of the structured matrices considered here were chosen for their importance in applications, the real enabling structure is that present in $H \otimes H$. This is especially illustrated by certain normal matrices (see definition 3.1).

The balance of this paper is organized as follows. In the next section some notation and one observation which is used throughout are recorded. In the same section the relation between $H \otimes H$ and $gl(4, R)$ is presented. The third section discusses a wide family of matrices which can be exponentiated using the aforementioned algebra isomorphism, and contains an illustrative example. The final section offers conclusions. In the first appendix, the second approach to exponentiating matrices in $p(4, R)$ and $so(2, 2, R)$ (see the next section for notation) is presented in a manner that makes the connection to the covering space homomorphism between $SU(2) \times SU(2)$ and $SO(4, R)$ explicit (see remark 5.1). In the final section, 13 classes of matrices are listed which can be exponentiated by mimicking verbatim two situations studied earlier.

In closing this introductory section it is noted that by combining these techniques with techniques such as structure preserving similarities, e.g., Givens rotations [12], one can extend these results, in many cases, to find algorithmically the exponentials of structured matrices of size *bigger than* 4. In other words, one can use such similarities (normally used in the literature for reduction to canonical forms) to reduce the exponential calculation to dimension four or lower of matrices with similar structure. In principle, this would provide closed form formulae for the exponentials of such structured matrices, since one can explicitly write the desired Givens type similarities. However, it is more accurate to say that this implies an algorithmic procedure for exponentiating such matrices. For matrices for which this is possible (e.g., symmetric matrices), the details of this procedure are routine and hence will not be pursued here.

2. Notation and preliminary observations

The following definitions and notations will be frequently met in this work:

- $gl(n, R)$ and $gl(n, C)$ represent the algebra of real (respectively complex) $n \times n$ matrices. These are, of course, the Lie algebras of the Lie groups of real (complex) invertible matrices.
- $sl(n, R)$ and $sl(n, C)$ represent the algebra of real (respectively complex) traceless matrices. $SL(n, R)$ and $SL(n, C)$ represent the corresponding Lie groups of real (respectively complex) matrices of determinant 1.
- $SU(n)$ represents the Lie group of $n \times n$ unitary matrices of determinant 1. $su(n)$ represents the corresponding Lie algebra of $n \times n$ skew-Hermitian, traceless matrices. Note it is customary to use the terminology 'anti-Hermitian' for skew-Hermitian matrices.
- R_n represents the matrix with 1 on the anti-diagonal and zeroes elsewhere. $p(n, R)$ and $p(n, C)$ represent the algebra of $n \times n$ real (respectively complex) matrices, A, satisfying $A^T R_n + R_n A = 0$. These matrices are also said to be *perskewsymmetric*. *Persymmetric* matrices are those matrices, X, which satisfy $X^T R_n = R_n X$. Such matrices are symmetric about the anti-diagonal. $P(n, R)$ (respectively $P(n, C)$) is the corresponding Lie group, namely, the set of matrices (real/complex), X, which satisfy $X^T R_n X = R_n$.
- J_{2n} is the $2n \times 2n$ matrix which, in block form, is given by $J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. $sp(2n, R)$ and $sp(2n, C)$ represent the Lie algebra of those real (respectively complex) $2n \times 2n$ matrices which satisfy $X^T J_{2n} + J_{2n} X = 0$. Such matrices are also called *Hamiltonian*. Matrices, Z, satisfying $Z^T J_{2n} = J_{2n} Z$ are called *skew-Hamiltonian* (sometimes anti-Hamiltonian).
- $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. $so(p, q, R)$ and $so(p, q, C)$ represent the Lie algebra of real (respectively complex) $n \times n$ matrices ($n = p + q$), X, satisfying $X^T I_{p,q} + I_{p,q} X = 0$.
- The *anti-trace* of an $n \times n$ matrix is the sum of the elements on its anti-diagonal. X, $n \times n$, is *anti-scalar* if $X = \gamma R_n$, with $\gamma \in R$ (or C).
- Throughout H will be denote the field (more precisely the division algebra) of the *quaternions*, while P stands for the *purely imaginary* quaternions, tacitly identified with R^3.

Remark 2.1.

(i) Throughout this paper, use of the following observation will be made: let X be an $n \times n$ matrix satisfying $X^2 + c^2 I_n = 0$, $c \neq 0$. Then $e^X = \cos(c) I_n + \frac{\sin(c)}{c} X$. Here c^2 is allowed to be complex, and c is then taken to be $\sqrt{r}\, e^{i\frac{\theta}{2}}$, with $c^2 = r\, e^{i\theta}$, $\theta \in [0, 2\pi)$. Note, in particular, that if c^2 is purely imaginary (equivalently, if $X^2 = d^2 I_n$, with d real), then the cos and sin in the formula for e^X become their hyperbolic equivalents, cosh and sinh respectively.

(ii) Occasionally the fact that any matrix which satisfies $X^3 = -c^2X, c \neq 0$, satisfies $e^X = I + \frac{\sin(c)}{c}X + \frac{1-\cos(c)}{c}X^2$ (Rodrigues's formula) will also be used. Once again c^2 is permitted to be complex.

(iii) Explicit formulae for e^A can be produced if the minimal polynomial of A is known and it is low in degree (cf [13] where such formulae are written from the characteristic polynomial). Since it is possible to find the minimal polynomial of many of the matrices considered here explicitly (i.e., without any spectral information), this removes the need for the spectral decomposition, mentioned in the introduction, for A symmetric. However, since the corresponding explicit formulae for e^A are more complicated than those in (i) and (ii), they will not be pursued here. See the conclusions for an illustration of this issue.

$H \otimes H$ **and** $gl(4, R)$: the algebra isomorphism between $H \otimes H$ and $gl(4, R)$, which is central to this work is the following:

- Associate with each product tensor $p \otimes q \in H \otimes H$, the matrix, $M_{p\otimes q}$, of the map which sends $x \in H$ to $px\bar{q}$, identifying R^4 with H via the basis $\{1, i, j, k\}$. Thus, if $p = p_0 + p_1i + p_2j + p_3k$; $q = q_0 + q_1i + q_2j + q_3k$, then

$$M_{p\otimes q} = \begin{pmatrix} u_0 & v_0 & w_0 & z_0 \\ u_1 & v_1 & w_1 & z_1 \\ u_2 & v_2 & w_2 & z_2 \\ u_3 & v_3 & w_3 & z_3 \end{pmatrix}$$

  with

$$\begin{aligned} p\bar{q} &= u_0 + u_1i + u_2j + u_3k \\ pi\bar{q} &= v_0 + v_1i + v_2j + v_3k \\ pj\bar{q} &= w_0 + w_1i + w_2j + w_3k \\ pk\bar{q} &= z_0 + z_1i + z_2j + z_3k. \end{aligned}$$

  Here, $\bar{q} = q_0 - q_1i - q_2j - q_3k$.
- Extend this to the full tensor product by linearity, e.g., the matrix associated with $2(p_1 \otimes q_1) - 9(p_2 \otimes q_2)$ is the matrix $2M_{p_1\otimes q_1} - 9M_{p_2\otimes q_2}$. This yields an algebra isomorphism between $H \otimes H$ and $gl(4, R)$. In particular, a basis for $gl(4, R)$ is provided by the 16 matrices $M_{e_x\otimes e_y}$ as e_x, e_y run through $1, i, j, k$. In particular, $R_4 = M_{j\otimes i}$, $J_4 = M_{1\otimes j}$ belong to this basis.

This connection, which is known from the theory of Clifford algebras, has been put to great practical use in solving eigenvalue problems for structured matrices by Mackey *et al*, [9–11]. It can also be used for finding exponentials, $e^A, A \in gl(4, R)$ via the following procedure:

General algorithm for e^A *using* $H \otimes H$

(i) Identify $u \in H \otimes H$, corresponding to A via this isomorphism.
(ii) Find $e^u \in H \otimes H$ (in general, this will be possible in closed form only if u (and, hence A) possesses additional structure).
(iii) Find the matrix M corresponding to e^u—this is e^A.

Note. Throughout this work, tacit use of $H \otimes H$ representations of matrices in $gl(4, R)$ will be made. These can be easily obtained from the entries of the 4 × 4 matrix in question (see [9–11] for some instances). The key to this consists of the following two observations:

- Conjugation in $H \otimes H$ corresponds to matrix transposition in $gl(4, R)$, i.e., $M_{\bar{p}\otimes\bar{q}} = (M_{p\otimes q})^T$. This is why, for instance, symmetric matrices correspond to $c(1 \otimes 1) + p \otimes i + q \otimes j + r \otimes k$ with p, q, r purely imaginary, and skew-symmetric matrices correspond to $s \otimes 1 + 1 \otimes t$, $s, t \in P$.
- Hamiltonian (respectively skew-Hamiltonian) matrices are expressible as $J_{2n}S$, with S symmetric (respectively skew-symmetric). Similarly persymmetric (respectively perskewsymmetric) matrices are expressible as $R_n S$ with S symmetric (respectively skew-symmetric). Thus, for instance, perskewsymmetric matrices are represented by $(j \otimes i)[s \otimes 1 + 1 \otimes t]$, $s, t \in P$. This simplifies to $p \otimes i + \alpha(j \otimes 1) + j \otimes q + \beta(1 \otimes i)$ with $p \in \text{span}\{i, k\}$, $q \in \text{span}\{j, k\}$, $\alpha, \beta \in R$. If such a matrix is simultaneously symmetric, then $\alpha = \beta = 0$, etc.

Combining these two observations with the explicit forms of the 16 matrices, $M_{e_x \otimes e_y}$ (see, e.g., [9] for these explicit forms) leads to $H \otimes H$ representations, in terms of the entries of the matrices.

3. Exponentials of structured 4 × 4 matrices

In this section, the algebra isomorphism between $H \otimes H$ and real 4×4 matrices will be used to find exponentials of various structured matrices. For many of these matrices, their exponentials can be found directly from their $H \otimes H$ representations. These will be presented first. For the remaining the singular value factorization of matrices, no bigger than 3×3, is needed. This can be done in closed form [15]. These will be presented next.

3.1. Exponentials directly from $H \otimes H$ representation

Below a (by no means exhaustive) list of nine families of real 4×4 matrices, whose exponentials can be directly found from their $H \otimes H$ representations, is presented. These families seem to be ubiquitous in applications.

1. 4×4 *skew-symmetric matrices*. The corresponding element in $H \otimes H$ is $p \otimes 1 + 1 \otimes q$ with $p, q \in P$. For finding its exponential, it is noted that $p \otimes 1$ and $1 \otimes q$ commute, so the exponential of the sum is the product of the individual exponentials. Now consider, $(p \otimes 1)^2 = -(\|p\|^2)1 \otimes 1$. Thus, $p \otimes 1$ is annihilated by a quadratic polynomial, and its exponential, therefore, is $\left[\cos(\|p\|)1 + \frac{\sin(\|p\|)}{(\|p\|)} p\right] \otimes 1 = x \otimes 1$, with x being a unit quaternion. Likewise, $\mathrm{e}^{(1\otimes q)} = 1 \otimes y$, y being a unit quaternion. Thus, e^A is the matrix $M_{x\otimes y}$, which is a different perspective on the $SU(2) \times SU(2)$, $SO(4, R)$ relation.
2. 4×4 *perskewsymmetric matrices*. Such matrices P have $H \otimes H$ representations $p \otimes i + \alpha(j \otimes 1) + j \otimes q + \beta(1 \otimes i) = X + Y + Z + W$ with $p \in \text{span}\{i, k\}$, $q \in \text{span}\{j, k\}$, $\alpha, \beta \in R$, we find X, Y both commute with each of Z and W. Hence $\mathrm{e}^P = \mathrm{e}^{(X+Y)}\, \mathrm{e}^{(Z+W)}$. Further, $XY = -YX$, $ZW = -WZ$.

   Next, since $(X+Y)^2 = (\|p\|^2 + \alpha^2)1 \otimes 1$, $\mathrm{e}^{(X+Y)} = \cosh(\lambda_1)(1 \otimes 1) + \frac{\sinh(\lambda_1)}{\lambda_1}(X+Y) = \cosh(\lambda_1)(1 \otimes 1) + \frac{\sinh(\lambda_1)}{\lambda_1}(p \otimes i + \alpha(j \otimes 1))$, with $\lambda_1 = \sqrt{(\|p\|^2 + \alpha^2)}$. Likewise, $\mathrm{e}^{(Z+W)} = \cosh(\lambda_2)(1 \otimes 1) + \frac{\sin(\lambda_2)}{\lambda_2}(Z + W) = \cosh(\lambda_2)(1 \otimes 1) + \frac{\sinh(\lambda_2)}{\lambda_2}(j \otimes q + \beta(1 \otimes i))$, with $\lambda_2 = \sqrt{(\|q\|^2 + \beta^2)}$.

   Hence, e^P is the matrix representation of $\left\{\cosh(\lambda_1)(1 \otimes 1) + \frac{\sinh(\lambda_1)}{\lambda_1}[(p \otimes i) + \alpha(j \otimes 1)]\right\}$ $\left\{\cosh(\lambda_2)(1 \otimes 1) + \frac{\sinh(\lambda_2)}{\lambda_2}[(j \otimes q + \beta(1 \otimes i)]\right\}$.
3. 4×4 *skew-Hamiltonian matrices*. Such matrices, S, have $H \otimes H$ representations of the form $b(1 \otimes 1) + p \otimes j + 1 \otimes (ci + dk)$, with $b, c, d \in R$ and $p \in P$. Clearly the $b(1 \otimes 1)$

component commutes with the remaining summands. Thus $e^S = e^b \exp(p \otimes j + 1 \otimes (ci + dk))$. Now, $(p \otimes j + 1 \otimes (ci + dk))^2 = -(-\|p\|^2 + c^2 + d^2)(1 \otimes 1)$. Indeed the two summands anti-commute, while $(p \otimes j)^2 = \|p\|^2(1 \otimes 1)$; $(ci + dk)^2 = -(c^2 + d^2)(1 \otimes 1)$. Hence $e^S = e^b\big(\cos(\lambda)(1 \otimes 1) + \frac{\sin(\lambda)}{\lambda}(p \otimes j + 1 \otimes (ci + dk))\big)$, with $\lambda = \sqrt{(-\|p\|^2 + c^2 + d^2)}$. Note $\lambda \in C$.

4. *Five Jordan algebras.* See appendix B.
5. *Eight Lie algebras.* See appendix B. In particular, one member of this list is precisely $so(2, 2, R)$.
6. *Simultaneously Hamiltonian, symmetric, persymmetric matrices.* These have $H \otimes H$ representations of the form $M = X + Y + Z = \beta(j \otimes i) + \gamma(i \otimes k) + \delta(k \otimes k)$, $\beta, \gamma, \delta \in R$. Now X commutes with both Y, Z while Y, Z anti-commute, and each of X, Y, Z squares to a positive constant times $1 \otimes 1$. Hence e^M is the matrix representation of
$$e^X e^{(Y+Z)} = [\cosh(\beta)(1 \otimes 1) + \sinh(\beta)(j \otimes i)] \times \left[\cosh(\lambda)(1 \otimes 1) + \frac{\sinh(\lambda)}{\lambda}(\gamma(i \otimes k) + \delta(k \otimes k)\right], \qquad \lambda = \sqrt{\gamma^2 + \delta^2}.$$
7. *Some symmetric Toeplitz matrices.* The general case of a symmetric, Toeplitz matrix is subsumed by the case of bisymmetric matrices, see remark 3.3. Here we identify two important classes which do not require the intervening spectral factorization calculations for the general case.
   - *Symmetric, Toeplitz, tridiagonal matrix.* Since such a matrix is met frequently in applications, it worth noting that its exponential can be directly computed in closed form. Indeed, their $H \otimes H$ representations are given by $a(1 \otimes 1) + \frac{b}{2}(j \otimes i) + \frac{b}{2}(i \otimes j) + b(k \otimes j), a, b \in R$. Expressing this as $X + Y + Z + W$, we see X and Y commute with both Z, W and further $XY = YX, ZW = -WZ$. Hence $e^{(X+Y+Z+W)} = e^X e^Y e^{(Z+W)} = e^a\big[\cosh\left(\frac{b}{2}\right)(1 \otimes 1) + \sinh\left(\frac{b}{2}\right)(j \otimes i)\big]\big[\cosh(c)(1 \otimes 1) + \frac{\sinh(c)}{c}((i + k) \otimes j)\big], c = \frac{\sqrt{5}}{4}b$.
   - *Symmetric, Toeplitz matrix S satisfying* $s_{13} = 0$. This implies that the *second superdiagonal and subdiagonal vanish*. Such matrices have $H \otimes H$ representations of the form $a(1 \otimes 1) + b(j \otimes i) + c(i \otimes j) + b(k \otimes j)$. Now, the first and second summand commute amongst themselves and with the remaining summands. While the third and the fourth anti-commute. Hence,
$$e^S = e^a[\cosh(b)(1 \otimes 1) + \sinh(b)(j \otimes i)] \times \left[\cosh(\lambda)(1 \otimes 1) + \frac{\sinh(\lambda)}{\lambda}(c(i \otimes j) + b(k \otimes j))\right], \qquad \lambda = \sqrt{b^2 + c^2}.$$
8. *Certain normal matrices.* The general case of normal matrices is subsumed by the algorithm below for a symmetric matrix, since the case of skew-symmetric matrices has already been dealt with (a matrix is normal iff its symmetric and skew-symmetric parts commute). Here we discuss a subclass which does not require the spectral factorization calculations needed for exponentiating a symmetric matrix. This subclass is described via the following:

Definition 3.1. *Consider a normal* $N = S + T$, *with* S *being its symmetric part and* T *being its skew-symmetric part. Expressing* T *as the sum of two commuting skew-symmetric matrices,* $T_1 = M_{s\otimes 1}$ *and* $T_2 = M_{1\otimes t}$, $s, t \in P$ *it is assumed that* $\|s\| \neq \|t\|$. *Such matrices will be called special normal.*

Note that special normality forces $T \neq 0$. Special normality also implies that $[S, T_i] = 0, i = 1, 2$ (this will be shown below). It is this condition that makes exponentiation in closed form possible.

Indeed, consider, first the case that $T_1 \neq 0$. Letting $S = a(1 \otimes 1) + p \otimes i + q \otimes j + r \otimes k$, the assumption $[S, T_1] = 0$ forces, in conjunction with the linear independence of the elements $e_x \otimes e_y, e_x, e_y = i, j, k$, *each* of the $p \otimes i, q \otimes j, r \otimes k$ to commute with $s \otimes 1$. This implies that each of p, q, r is parallel to s and hence the symmetric part of N can be expressed succinctly as

$$a(1 \otimes 1) + s \otimes \hat{t}, s, \hat{t} \in P.$$

Now the condition, $[S, T_2] = 0$ forces t to be parallel to $\hat{t}$. Hence we find

$$\mathrm{e}^N = \mathrm{e}^a \left[\cosh(\lambda) I_2 + \frac{\sinh(\lambda)}{\lambda} (s \otimes \hat{t}) \right] (\mathrm{e}^s \otimes 1)(1 \otimes \mathrm{e}^t), \lambda = \|s\| \|\hat{t}\|.$$

If $T_1 = 0$, then the condition $[S, T_2] = 0$ implies that each of p, q, r is parallel to one another (w.l.o.g $p \neq 0$), and hence $S = p \otimes \hat{t}$, with $\hat{t} = kt, k \in R$. Hence the above formula holds with minor modification.

Next, it will be shown that special normality implies $[S, T_i] = 0, i = 1, 2$. One first shows

$$T^4 + 2(\|s\|^2 + \|t\|^2) T^2 + [(\|s\|^2 - \|t\|^2)^2] I = 0. \tag{3.1}$$

The calculation leading to the above simultaneously shows (i) T's minimal polynomial is quadratic iff either of s or t vanishes (in this case, trivially $[S, T_i] = 0, i = 1, 2$; (ii) T's minimal polynomial is cubic iff $\|s\| = \|t\|$. Hence, w.l.o.g T's minimal polynomial is quartic, i.e., T is *non-derogatory*.

Next, since S, T commute, they are simultaneously diagonalizable, via some unitary matrix U. Consider $U^* T U = U^* T_1 U + U^* T_2 U$. The last two matrices commute (since T_1, T_2 commute) and their sum is diagonal. If the entries of the diagonal matrix $U^* T U$ are all distinct and non-zero, then the matrices $U^* T_1 U$ and $U^* T_2 U$ are themselves diagonal. Thus they also commute with $U^* S U$, which implies that S commutes with both the T_i. Note T being non-derogatory implies the assumptions about the diagonal entries of $U^* T U$, in view of the nature of the eigenvalues of a 4×4 skew-symmetric matrix.

9. *Certain non-Toeplitz bisymmetric matrices.* Every persymmetric matrix is of the form RS, with S being symmetric. Similarly, Hamiltonian matrices are of the form JS, with S being symmetric. *Such matrices can often be exponentiated in closed form, if in addition,* $R_4 S = S R_4$ *(respectively* $J_4 S = S J_4$*).*

   Indeed, since $RS = SR$, and $R^2 = I$, we find $\mathrm{e}^{RS} = \cosh(S) + R \sinh(S)$ (this equation is valid in any dimension). Now, $S = a(1 \otimes 1) + p \otimes i + q \otimes j + r \otimes k$ satisfies $R_4 S = S R_4$ iff (i) p is parallel to j and (ii) q, r are perpendicular to j. If, in addition we suppose either q, r are parallel or q, r are perpendicular to one another, then exponentiation in closed form is possible. For brevity the former possibility is assumed. Hence

$$S = a(1 \otimes 1) + \epsilon(j \otimes i) + (\alpha i + \beta k) \otimes (\gamma j + \delta k).$$

   Note, in particular, that RS is symmetric, persymmetric, *but not Toeplitz*.

   Writing RS as $R(\mu I_4 + \tilde{S})$, with $\tilde{S} = X + Y$, we see that it suffices to find $\mathrm{e}^{R\tilde{S}}$. Now, note that X and Y commute and $X^2 = \epsilon^2 I, Y^2 = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) I = \lambda^2 I$. Hence, $\cosh(\tilde{S}) = \cosh(X)\cosh(Y) + \sinh(X)\sinh(Y)$, and $\sinh(\tilde{S}) = \sinh(X)\cosh(Y) + \sinh(Y)\cosh(X)$. But $\sinh(X) = \frac{\sinh(\epsilon)}{\epsilon} X$; $\sinh(Y) = \frac{\sinh(\lambda)}{\lambda} Y$; $\cosh(X) = \cosh(\epsilon) I$;

$\cosh(Y) = \cosh(\lambda)I$. Hence e^{RS} is the matrix given by

$$\left[\cosh(\mu)I_4 + \frac{\sinh(\mu)}{\mu}R\right]\left[\cosh(\epsilon)\cosh(\lambda)I + \frac{\sinh(\epsilon)\sinh(\lambda)}{\lambda\epsilon}XY\right.$$
$$\left. + R\left[\frac{\sinh(\epsilon)\cosh(\lambda)}{\epsilon}X + \frac{\sinh(\lambda)\cosh(\epsilon)}{\lambda}Y\right]\right].$$

Similarly, if $JS = SJ$, one finds (since $J^2 = -I$) that

$$e^{J_{2n}S} = \cos(S) + J_{2n}\sin(S).$$

Now if S, symmetric, commutes with J, then fortunately (or unfortunately) J_4S is also simultaneously skew-symmetric, and therefore the previous formula is yet another way of exponentiating J_4S. Hence, the details are omitted.

3.2. *The general symmetric case*

Exponentiating the general 4×4 symmetric matrix requires the spectral factorization of a 3×3 matrix (which can be done in closed form). Before getting to that, the principal enabling feature of the algorithm below is described by the following:

Proposition 3.1. *The exponential of* $a(1 \otimes 1) + \sum_{i=1}^{3} u_i \otimes v_i, u_i, v_i \in P$, *with* $\{u_i, i = 1, \ldots, 3\}, \{v_i, i = 1, \ldots, 3\}$ *each an orthogonal triple in* R^3 *is given by* $e^a \Pi_{i=1}^{3} e^{(u_i \otimes v_i)}$, *with* $e^{(u_i\otimes v_i)} = \cosh(\|u_i\|\|v_i\|)(1 \otimes 1) + \frac{\sinh(\|u_i\|\|v_i\|)}{(\|u_i\|\|v_i\|)}(u_i \otimes v_i)$.

Proof. It suffices to observe that each of the summands in $a(1 \otimes 1) + \sum_{i=1}^{3} u_i \otimes v_i$ commutes with each other due to the orthogonality property. The formula for $e^{(u_i\otimes v_i)}$ is now just a consequence of $(u_i \otimes v_i)$ squaring to a positive constant times the identity. □

Remark 3.1. If the triples $\{u_i\}, \{v_i\}$ were instead each parallel to each other, then once again the exponential of $a(1 \otimes 1) + \sum_{i=1}^{3} u_i \otimes v_i$ is quickly computed, since now once again each summand commutes with one another. There are other possible configurations which will render the calculation of the exponential in closed form too. However, these will not be pursued here for brevity.

Remark 3.2.

(i) Consider the element $p \otimes i + q \otimes j + r \otimes k, p, q, r \in P$. Then, as observed in [9], if $\sum_{i=1}^{3} \sigma_i u_i v_i^T, u_i, v_i \in R^3$ is the singular value factorization of the real 3×3 matrix, $[p \mid q \mid r]$, where the σ_i are the singular values and the u_i, v_i the left and right singular vectors, it follows that $p \otimes i + q \otimes j + r \otimes k = \sum_{i=1}^{3} \mu_i u_i \otimes v_i$, where the vectors u_i, v_i have been identified with the corresponding pure quaternions (in lieu of the elegant proof in [9], one can also verify this via direct calculations which show that if for $p_i, q_i, r_i, s_i \in P, i = 1, \ldots, 3$, the 3×3 matrices $\sum_{i=1}^{3} p_i q_i^T, \sum_{i=1}^{3} r_i s_i^T$ coincide, then $\sum_{i=1}^{3} M_{p_i\otimes q_i} = \sum_{i=1}^{3} M_{r_i\otimes s_i}$). Since the $\{u_i\}, \{v_i\}$ are each an orthonormal triple, the exponential of $p \otimes i + q \otimes j + r \otimes k$, which equals the exponential of $\sum_{i=1}^{3} \sigma_i u_i \otimes v_i$, can be explicitly found by using proposition 3.1. The only issue is computing the singular value factorization of a real 3×3 matrix. However, this is the spectral factorization of a real 3×3 symmetric matrix, which itself can be done in closed form [15]. It is interesting to note that the technique described in [15], consisting of 3×3 matrix manipulations, can itself be implemented via quaternions.

(ii) Note the subsequent rotations employed in [9] to diagonalize a symmetric matrix are *not required*, since diagonalization is not being employed here to compute exponentials. Only the reduction to form used in proposition 3.1 is needed.

This leads to the following algorithm for the *exponential of a* 4×4 *symmetric matrix*:

- Represent the matrix as $a(1 \otimes 1) + p \otimes i + q \otimes j + r \otimes k$, $p, q, r \in P$.
- Compute the singular value factorization, $\sum_{i=1}^{3} \sigma_i u_i v_i^T$, $u_i, v_i \in R^3$ of the real 3×3 matrix, $[p \mid q \mid r]$.
- Compute the exponential of $a(1 \otimes 1) + \sum_{i=1}^{3} \sigma_i u_i \otimes v_i$ via proposition 3.1. The 4×4 matrix representing this element of $H \otimes H$ is e^A.

Remark 3.3. The special classes of 4×4 bisymmetric matrices (i.e., simultaneously symmetric and persymmetric) and 4×4 symmetric and Hamiltonian matrices are, of course, subsumed by the foregoing algorithm. However, it is worth pointing out, in view of their importance in applications, that the singular value factorization needed is easier to find than in the fully symmetric case. Indeed, a bisymmetric matrix is represented by $a(1 \otimes 1) + b(j \otimes i) + p \otimes j + q \otimes k$, $p, q \in \operatorname{span}\{i, k\}$, $a, b \in R$. Thus, it suffices to find the singular value factorization of the 2×2 matrix $[p \mid q]$, which is the spectral factorization of a 2×2 real symmetric matrix. Likewise, a symmetric, Hamiltonian matrix is represented by $q \otimes i + r \otimes k$. Thus, it suffices to find the singular value factorization of the 3×2 matrix $[p \mid q]$. As will be seen in the example in this section, only two of the left singular vectors are needed, and hence once again, only a 2×2 matrix calculation is all that is needed. There are many other cases of symmetric matrices possessing additional symmetry which are susceptible to the same observation.

Remark 3.4. *Extension to complex matrices.* Some of the procedures extend to special classes of complex matrices. This is illustrated for matrices in $so(4, C)$. Such a matrix can be represented in the form $\alpha_1 M_{i\otimes 1} + \beta_1 M_{j\otimes 1} + \gamma_1 M_{k\otimes 1} + \alpha_2 M_{1\otimes i} + \beta_2 M_{1\otimes j} + \gamma_2 M_{1\otimes k} = \sum_{l=1} X_l$, with $\alpha_i, \beta_i, \gamma_i \in C$. Now the fact that these constants are complex does not prevent $X_i X_j = X_j X_i$, $i = 1, \ldots, 3$; $j = 4, \ldots, 6$. Neither does it prevent $X_k X_l + X_l X_k = 0$, $l, k = 1, \ldots, 3$, nor does it prevent $X_k X_l + X_l X_k = 0$, $l, k = 4, \ldots, 6$. Finally, $X_i^2 = -c_i^2 I_4$ for each $i = 1, \ldots, 6$, for some $c_i \in C$. Hence the exponential is given by

$$\left[\cos(\lambda_1) I_4 + \frac{\sin(\lambda_1)}{\lambda_1}(\alpha_1 M_{i\otimes 1} + \beta_1 M_{j\otimes 1} + \gamma_1 M_{k\otimes 1})\right] \times \left[\cos(\lambda_2) I_4 + \frac{\sin(\lambda_2)}{\lambda_2}(\alpha_2 M_{1\otimes i} + \beta_2 M_{1\otimes j} + \gamma_2 M_{1\otimes k})\right]$$

with $\lambda_i^2 = -\left(\alpha_i^2 + \beta_i^2 + \gamma_i^2\right)$, $i = 1, 2$. Similarly the technique for $p(4, R)$ extends verbatim to $p(4, C)$. However, while the methods based on the singular value factorization extend verbatim for purely imaginary symmetric matrices, they are not applicable to general complex symmetric matrices. To see what is needed for the extension, consider traceless symmetric matrices (w.l.o.g). Let A_R and A_I be the real and imaginary parts of A. Since these are also symmetric, one can associate two triples $(p_i, q_i, r_i) \in P^3$, $i = 1, 2$. Let $M_i = [p_i \mid q_i \mid r_i]$ be the corresponding real 3×3 matrices. If these could be simultaneously brought into the canonical forms $M_i = \sum_{k=1}^{3} \sigma_k^i u_k v_k^T$, with the u_k and v_k orthonormal, $\sigma_k^i \in R$, then clearly the algorithm for real symmetric matrices would extend verbatim to such matrices. Many sufficient conditions are known for such simultaneous canonical form [16]. One such condition is that both $M_1 M_2^T$, $M_2^T M_1$ should be symmetric.

3.3. Illustrative example

In this section we illustrate the above results for finding e^H, when H is a 4×4 Hamiltonian and symmetric matrix. A concrete situation where this problem arises is that of parametrizing all squeezing transformations for two mode quantum states. The authors of [6] have argued that the most general such operation is given by the unitary metaplectic representation of a symplectic matrix which is also positive definite. A matrix is symplectic if $X^T J_{2n} X = J_{2n}$. Such matrices form the Lie group corresponding to the Lie algebra of Hamiltonian matrices.

Now any symplectic matrix may be factorized as $X = PO$, where P is positive definite and O is orthogonal—this is just the polar decomposition of X. It is known that both P and O are also simultaneously symplectic [6].

Denoting by U_X the operator given by the metaplectic representation of a symplectic X, it is shown in [6] that operators of the form U_O conserve the total photon number, while those of the form U_P do not. Thus, parametrizing all squeezing operations requires, as a preliminary step, parametrizing all symplectic positive definite matrices. The second step of finding the metaplectic representation is standard (it consists of a sequence of Fourier transforms) and it is not the purpose of this work.

Remark 3.5. It is customary to write the operators, U_X, in the form $\exp(\mathrm{i}L)$, where L is an operator which is (typically) quadratic in the creation and annihilation operators. However, as it stands, this is just a formal expression. Since the collection of such operators forms a Lie algebra, isomorphic to $sp(2n, R)$, the proper interpretation of this expression is that $\exp(\mathrm{i}L) = U_S$, with $S = e^H$, where $H \in sp(2n, R)$ maps to L under this isomorphism. However, this suggestive notation is an expression of the fact that the collection of metaplectic representations of all symplectic matrices forms a group—the metaplectic group—which is a two-one cover of the symplectic group, just as $SU(2)$ is a two-one cover of $SO(3, R)$. However, unlike $SU(2)$, the metaplectic group is not a group of matrices.

Now every positive definite, symplectic matrix is the exponential of a symmetric matrix, which is simultaneously Hamiltonian. Thus, for two mode systems, this preliminary step can be achieved by exponentiating all symmetric, Hamiltonian 4×4 matrices.

Thus, let H be a 4×4 symmetric, Hamiltonian matrix. Then $H = J_4 S$, where S is symmetric. Since H is also simultaneously symmetric, we find (by setting equal to zero the skew-symmetric part of H) that $H = M_{q\otimes i + r\otimes k}$, with $q, r \in P$. The explicit formulae for q, r in terms of the entries of H are easily found [10]. They are recorded here for completeness.

$$q = q_1 i + q_2 j + q_3 k = \frac{h_{11} + h_{22}}{2} i + h_{14} j + \frac{h_{24} - h_{13}}{2} k$$
$$r = r_1 i + r_2 j + r_3 k = \frac{h_{13} + h_{24}}{2} i - h_{12} j + \frac{h_{11} - h_{22}}{2} k.$$

The procedure for finding the exponential of H now is the following:

- Form the 3×2 matrix Y,

$$Y = \begin{pmatrix} q_1 & r_1 \\ q_2 & r_2 \\ q_3 & r_3 \end{pmatrix}.$$

- Find the singular values σ_1, σ_2 and the right singular vectors v_1, v_2 of Y. The former are the positive square roots of the eigenvalues of the 2×2 symmetric, positive definite matrix

$Y^TY = \begin{pmatrix} q^Tq & q^Tr \\ r^Tq & r^Tr \end{pmatrix}$, and the latter are orthonormal eigenvectors v_1, v_2 of Y^TY belonging to these two eigenvalues of Y^TY. Explicitly,

$$\sigma_1 = \sqrt{q^Tq\cos^2\theta + r^Tr\sin^2\theta - q^Tr\sin(2\theta)}$$
$$\sigma_2 = \sqrt{q^Tq\sin^2\theta + r^Tr\cos^2\theta + q^Tr\sin(2\theta)}$$
$$v_1 = (\cos\theta, -\sin\theta)^T \qquad v_2 = (\sin\theta, \cos\theta)^T$$

with $\tan(2\theta) = \frac{2q^Tr}{r^Tr - q^Tq}$. Denote by v_i also, the corresponding pure quaternions whose j-component is zero, i.e.,

$$v_1 = \cos\theta i - \sin\theta k \qquad v_2 = \sin\theta i + \cos\theta k.$$

- Find $u_i = Yv_i$, $i = 1, 2$. Thus,
$$u_1 = (q_1\cos\theta - r_1\sin\theta, q_2\cos\theta - r_2\sin\theta, q_2\cos\theta - r_2\sin\theta)^T$$
$$u_2 = (q_1\sin\theta + r_1\cos\theta, q_2\sin\theta + r_2\cos\theta, q_3\sin\theta + r_3\cos\theta)^T.$$
The u_i are almost the left singular vectors of Y. The only difference is that $\|u_i\| = \sigma_i$, instead of being 1. For the purpose at hand this difference is insignificant. The third left singular vector, corresponding to the 0 eigenvalue of YY^T, is *not* required. The usual singular value decomposition of Y now reads $Y = \sum_{i=1}^{2} u_i v_i^T$. Denote by u_i also, the corresponding pure quaternions. Then $H = \sum_{i=1}^{2} M_{u_i \otimes v_i}$.
- Find the elements, $w_i \in H \otimes H$, $i = 1, 2$ given by $w_i = \cosh(\sigma_i) 1 \otimes 1 + \frac{\sinh(\sigma_i)}{\sigma_i} u_i \otimes v_i$.
- Let M_i, $i = 1, 2$ be $\cosh(\sigma_i) I_4 + \frac{\sinh(\sigma_i)}{\sigma_i} M_{u_i \otimes v_i}$.
- Then $\mathrm{e}^H = M_1 M_2$.

Thus, e^H is parametrized by the six parameters $q_l, r_l, l = 1, \ldots, 3$, with the entries of e^H being some computable quantities of these six parameters. Note, since here the exponential is being used only for parametrization, one can begin directly with the $p_l, r_l, l = 1, \ldots, 3$, i.e., one need not write the p_l, r_l in terms of the h_{ij}.

4. Conclusions

In this paper, closed form formulae are provided for exponentials of several important families of real (and complex) 4×4 matrices. In conjunction, with techniques such as Givens rotations, these formulae provide algorithms for exponentiating classes of structured matrices in higher dimensions. The principal technique is the invocation of the associative algebra isomorphism between $gl(4, R)$ and $H \otimes H$. It is the ease of multiplication in $H \otimes H$ which facilitates the discovery of closed form exponentials for many matrices.

It is possible to write exponentials of matrices once their minimal polynomial is known (especially if they are at most quartic). However, these formulae themselves can be quite complicated and hence they were not pursued in this paper. This is exemplified by generic 4×4 skew-symmetric matrices, whose minimal polynomial is quartic. The corresponding exponential formula, though equivalent to the one given here, is substantially more complicated. In our opinion most 4×4 matrix calculations should be done in $H \otimes H$. The formulae for the minimal polynomial of a 4×4 skew-symmetric matrix (see equation (3.1)), without any spectral information, are yet another vivid illustration.

Clearly, $\tilde{H} \otimes \tilde{H}$ is associative algebra isomorphic to $gl(4, c)$, where $\tilde{H}$ is the complexification of H. One can identify the latter with $gl(2, C)$. However, it is better to view its elements as $q = x_0 + x_1 i + x_2 j + x_3 k$, $x_i \in C$ and define $\bar{q} = \bar{x}_0 - \bar{x}_1 i - \bar{x}_2 j - \bar{x}_3 k$. This notion of conjugation is equivalent to Hermitian conjugation in $gl(4, C)$. This does not, however, render calculating exponentials in $su(4)$ as simple as in $so(4, R)$ (after all one cannot run away from the curse of dimensionality by such an artifice). However, several Hermitian and skew-Hermitian matrices (e.g., whose real and imaginary parts come from special normal real matrices) are easily exponentiated.

Appendix A

In this appendix, a different approach to the exponentiation of matrices in $p(4, R)$, $so(2, 2, R)$, $p(3, R)$ is described, which reduces the problem to the exponentiation of 2×2 matrices (this is equally applicable to their complex counterparts). This is first illustrated for matrices in $so(3, R)$ and $so(4, R)$ since this should be reasonably well-known terrain. Attention, in particular, is drawn to remark 5.1, which provides the correct heuristics needed to generalize this to the matrices in $p(4, R)$, $so(2, 2, R)$, $p(3, R)$.

Notation. Throughout this appendix, $\sigma_x, \sigma_y, \sigma_z$ represent the usual Pauli matrices. E_{mn} represents the square matrix whose sole non-zero entry is a 1 in the (m, n) entry.

Consider an element $A \in so(3, R)$. Its exponential can be computed explicitly via the Rodrigues formula. The usual derivation of this relies on the fact that A satisfies

$$A^3 + \lambda^2 A = 0, \lambda \in R.$$

Any matrix which satisfies this equation will satisfy the Rodrigues formula. There is a equally well-known relation between $su(2)$ and $so(3, R)$. What is, perhaps, less appreciated is that this relation yields an explicit technique to find e^A, $A \in so(3, R)$. To describe this, fix a $G \in SU(2)$. Consider $V = \{A \mid A^* = A, \mathrm{Tr}(A) = 0\}$. $SU(2)$ acts via conjugation on elements $A \in V$, namely, $\phi_G(A) = GAG^{-1}$. It is well known that upon identifying V with R^3 through the basis $\{\sigma_k, k = x, y, z\}$, this action yields a proper rotation of R^3. Thus, we get a homomorphism, $\phi : SU(2) \to SO(3, R)$, which sends G to the matrix of ϕ_G with respect to the basis $\{\sigma_k, k = x, y, z\}$. This is a surjective, two-one, homomorphism. Linearizing this map, we get a Lie-algebra isomorphism $\psi : su(2) \to so(3, R)$, namely, $\psi(A)$ is the matrix of the linear map which sends $v \in V$ to $Av - vA$ *with respect to the* $\{\sigma_k, k = x, y, z\}$ *basis*, with $A \in su(2)$. This is a Lie-algebra isomorphism. From elementary considerations in Lie theory ψ and ϕ provide the following technique to find e^A, $A \in so(3, R)$:

(i) Find $B = \psi^{-1}(A)$ in $su(2)$
(ii) Compute $\mathrm{e}^B \in SU(2)$—this can be explicitly done since B satisfies the condition in (i) of remark 2.1.
(iii) Compute the matrix $\phi_{(\mathrm{e}^B)}$—this is e^A.

This is arguably easier to use than the Rodrigues formula (it is left to the reader to verify that the two result in the same formula). This is not to disparage the Rodrigues formula—it applies to situations where Lie theory would have no visible role. But the fact that a 3×3 exponential has been computed with a 2×2 calculation is significant. Similar and even better savings occur by such arguments.

Exponentials in $so(4, R)$. There is a well known two-one Lie group homomorphism denoted by $\phi : SU(2) \times SU(2) \to SO(4, R)$, given by the action of $SU(2) \times SU(2)$ on the vector space, V, of real linear combinations of $I_2, i\sigma_k, k = x, y, z$, namely, for fixed $G, H \in SU(2) \times SU(2)$, let $\phi_{G,H} V \to V$ be given by $\phi_{G,H}(X) = GXH^{-1}, X \in V$. Once again this is a proper rotation of R^4 (identified with V via this basis), and $\phi(G, H)$ is precisely the *matrix of this map with respect to this basis*. Linearizing this gives a Lie algebra isomorphism, $\psi : su(2) \times su(2) \to so(4, R)$ which sends $(X, Y) \in su(2) \times su(2)$ to the matrix of the map (with respect to the $I_2, i\sigma_k$ basis) which sends $Z \in V$ to $XZ - YZ$. This yields an algorithm to find e^A, $A \in so(4, R)$, *which reduces to finding two* 2×2 *exponentials in* $su(2)$—the statement of the algorithm is omitted (mimick the $p(4, R)$ algorithm given below).

The corresponding relations between $SL(2, C)$ (respectively $SL(2, C) \times SL(2, C)$) and $SO(3, C)$ (respectively $SO(4, C)$) once again reduce exponentiation of matrices in $so(3, C)$ and $so(4, C)$ to 2×2 calculations. Note that the fact that $SO(3, C)$ etc, are not compact does not matter for the veracity of this procedure. All that is needed for finding e^A is that the corresponding ϕ be a Lie group homomorphism (it need not even be surjective) and the corresponding ψ be a Lie algebra isomorphism.

Remark 5.1. Traditional proofs of the $SU(2)$ covering of $SO(3, R)$ proceed by (i) using $su(2)$ itself as the vector space V, and (ii) then, by constructing a bilinear form, $K(X, Y) = \mathrm{Tr}(ad\, X\, ad\, Y)$ on $su(2)$ and showing that this is preserved by the action of $SU(2)$. For our purposes it is more useful to proceed differently. On any (sub)space of 2×2 matrices, there are two obvious candidates for quadratic forms, namely, (i) $\mathrm{Tr}(X^2)$; and (ii) $\det(X)$. One is even led inexorably to these forms upon inspecting the forms of the maps ϕ used above for both $so(3, R)$ and $so(4, R)$. Polarizing these two leads to the following choices:

$$L_1(X, Y) = \mathrm{Tr}(XY) \tag{A1}$$

$$L_2(X, Y) = \det(X + Y) - \det(X) - \det(Y). \tag{A2}$$

It is easy to see that, with the choice of bases made in the derivation of the $so(3, R)$ (respectively $so(4, R)$) algorithms, the symmetric matrices representing these two forms are, up to a real constant, precisely I_3 (respectively I_4). This immediately shows that the matrix of the corresponding ϕ is orthogonal.

Remark 5.2. *Lorenz Lie algebra.* Here a different perspective on the work of [14] on the exponentials of matrices in $so(1, 3, R)$ is provided. Indeed, letting V be the R-linear span of $\{I_2, \sigma_x, \sigma_y, \sigma_z\}$ (i.e., V is the space of 2×2 Hermitian matrices), it is found that the matrix of $L_2(X, Y)$ is $2I_{1,3}$. If $SL(2, C)$ acts on V via $\phi_M(v) = MvM^*$, $v \in V$, $M \in SL(2, C)$, then $L_2(X, Y)$ is preserved and the matrix of ϕ_M in this basis is in the Lorenz group. Linearizing ϕ, we get a technique to find exponentials in $so(1, 3, R)$, cf [14].

Below the same thinking is used to compute exponentials in $p(4, R)$, $so(2, 2, R)$ and $p(3, R)$. The method can be applied to several other Lie algebras stemming from symmetric, non-degenerate, bilinear forms on R^4. However, we limit ourselves to these cases for brevity.

Exponentials in $p(4, R)$. Consider $gl(2, R)$, identified with R^4 via the basis, $\{E_{11}, E_{12}, -E_{21}, E_{22}\}$. Let $SL(2, R) \times SL(2, R)$ act on $gl(2, R)$, via $\phi_{G,H}(X) = GXH^{-1}$. This action leaves the bilinear form $L_2(X, Y)$ of equation (A2) invariant. Furthermore the symmetric matrix representing it, with respect to this basis, is R_4. Thus the matrix of $\phi_{G,H}$ is in $P(4, R)$. Linearizing this we get a Lie-algebra isomorphism (that this is a Lie-algebra homomorphism is standard—it is easily verified that it is an isomorphism): $\psi : sl(2, R) \times sl(2, R) \rightarrow p(4, R)$, which sends a pair $(g, h) \in sl(2, R) \times sl(2, R)$ to the matrix of the linear map $L_{g,h}(X) = gX - Xh$, $X \in gl(2, R)$ with respect to the $\{E_{11}, E_{12}, -E_{21}, E_{22}\}$ basis. This leads to the following algorithm to find e^A, $A \in p(4, R)$.
Algorithm for e^A, $A \in p(4, R)$:

(i) Find the pair $(g, h) = \psi^{-1}(A) \in sl(2, R) \times sl(2, R)$;
(ii) Find $G = e^g$, $H = e^h$. This is easily done since g, h satisfy the equation in remark 2.1 (i);
(iii) Find the matrix of $\phi_{G,H}$ with respect to the above basis. This is e^A.

Exponentials in $so(2, 2, R)$. Now identify $gl(2, R)$ with R^4 via the basis $\{I_2, E_{12} - E_{21}, \sigma_x, \sigma_z\}$. Then the matrix of $L_2(X, Y)$, with respect to this basis is precisely $I_{2,2}$. Let $SL(2, R) \times SL(2, R)$ act on $gl(2, R)$ via $\phi_{G,H}(V) = GVH^{-1}, G, H \in SL(2, R), V \in gl(2, R)$. Then $\phi_{G,H}$ preserves $L_2(X, Y)$ and hence its matrix, with respect to this basis, is in $SO(2, 2, R)$. Linearizing this action we get a Lie algebra isomorphism $\psi : sl(2, R) \times sl(2, R) \rightarrow so(2, 2, R)$, with $\psi(g, h)$ being the matrix of the linear map $\psi_{g,h}(v) = gv - vh, v \in gl(2, R)$ with respect to the same basis. This leads to an algorithm, similar to the previous one, for finding e^A, $A \in so(2, 2, R)$.

Exponentials in $p(3, R)$. Now identify R^3 with the real span of E_{12}, σ_x, E_{21}. This is $sl(2, R)$. Then the matrix of $L_1(X, Y)$, with respect to this basis, is, up to a constant, R_3. Let $SL(2, R)$ act on this copy of R^3 via $\phi_G(h) = GhG^{-1}$. This action preserves $L_1(X, Y)$. Thus, the matrix of ϕ_G is in $P(3, R)$ and the map $\phi : SL(2, R) \rightarrow P(3, R)$ is easily seen to be a Lie group homomorphism. Linearizing ϕ leads to a Lie algebra isomorphism $\psi : sl(2, R) \rightarrow p(3, R)$ which sends $h \in sl(2, R)$ to the matrix of the linear map, which sends $X \in sl(2, R)$ to $hX - Xh$ (identifying $sl(2, R)$ with R^3 via the above basis). This leads to an algorithm for finding e^A, $A \in p(3, R)$.

Remark 5.3. (i) The last calculation can be mimicked to find exponentials in $so(2, 1, R)$. Indeed, identify $sl(2, R)$ with R^3 via the basis $\{\sigma_x, \sigma_z, E_{12} - E_{21}\}$ and proceed verbatim as in the $p(3, R)$ case. (ii) All of the above calculations extend to find exponentials in $p(4, C)$ etc. The only difference is one works with complexifications of the various Lie algebras introduced before, i.e., $sl(2, C) \times sl(2, C)$ for $p(4, C)$ etc.

Appendix B

In this appendix are listed (i) five classes of matrices, each a Jordan algebra, which can be exponentiated by mimicking the technique for skew-Hamiltonian matrices; (ii) eight classes of matrices, each forming a Lie algebra, which can be exponentiated by mimicking the technique for perskewsymmetric matrices. In most cases the technique extends to their complex analogues (e.g., $so(2, 2, C)$), cf remark 3.4. In both lists, the $H \otimes H$ representation is provided. The interesting block structures of these matrices, which are easy to write, are omitted.

Remark 6.1. Let M_1, M_2 be two invertible, symmetric (respectively skew-symmetric) matrices, with the corresponding bilinear form on R^n denoted by $\langle\, ,\, \rangle_{M_1}, \langle\, ,\, \rangle_{M_2}$. The two forms are defined to be equivalent if there is a special orthogonal matrix G such that $G^T M_1 G = M_2$. If this is the case then the corresponding Jordan algebras, $J_i = \{X \mid X^T M_i = M_i X\}, i = 1, 2$ and the corresponding Lie algebras $L_i = \{X \mid X^T M_i = -M_i X\}, i = 1, 2$ are conjugate. Specifically $J_2 = G^T J_1 G, L_2 = G^T L_1 G$. Thus, if one knows exponentials of matrices in J_1 (respectively L_1), then one can find exponentials of matrices in J_2 (respectively L_2) provided G is explicitly described.

In the first list, the first two Jordan algebras pertain to bilinear forms which are equivalent to J_4, while all the matrices in the second list stem from symmetric forms equivalent to R_4. While it is possible to explicitly construct the corresponding G*, it is far easier to work with the matrices in these lists directly*.

Exponentials of five Jordan algebras

- $p \otimes k + a(1 \otimes 1) + 1 \otimes (bi + cj), a, b, c \in R, p \in P$.
- $p \otimes i + a(1 \otimes 1) + 1 \otimes (bj + ck), a, b, c \in R, p \in P$.

- $i \otimes q + a(1 \otimes 1) + (bj + ck) \otimes 1$, $a, b, c \in R$, $q \in P$.
- $j \otimes q + a(1 \otimes 1) + (bi + ck) \otimes 1$, $a, b, c \in R$, $q \in P$.
- $k \otimes q + a(1 \otimes 1) + (bi + cj) \otimes 1$, $a, b, c \in R$, $q \in P$.

Exponentials of eight Lie algebras

- $so(2, 2, R)$. The $H \otimes H$ representation is $a(1 \otimes i) + i \otimes p + b(i \otimes 1) + q \otimes i$, $p, q \in \mathrm{span}\{j, k\}$, $a, b \in R$. The block representation is $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, where B is any 2×2 matrix, while A, C are 2×2 anti-diagonal matrices with zero anti-trace.
- $p \otimes j + a(j \otimes 1) + j \otimes q + b(1 \otimes j)$, $p, q \in \mathrm{span}\{i, k\}$, $a, b \in R$.
- $p \otimes k + a(k \otimes 1) + k \otimes q + b(1 \otimes k)$, $p, q \in \mathrm{span}\{i, j\}$, $a, b \in R$.
- $p \otimes i + a(k \otimes 1) + k \otimes q + b(1 \otimes i)$, $p \in \mathrm{span}\{i, j\}$, $q \in \mathrm{span}\{j, k\}$, $a, b \in R$.
- $p \otimes j + a(k \otimes 1) + k \otimes q + b(1 \otimes j)$, $p \in \mathrm{span}\{i, j\}$, $q \in \mathrm{span}\{i, k\}$, $a, b \in R$.
- $p \otimes j + b(i \otimes 1) + a(1 \otimes j) + i \otimes q$, $p \in \mathrm{span}\{j, k\}$, $q \in \mathrm{span}\{i, k\}$, $a, b \in R$.
- $p \otimes k + a(i \otimes 1) + b(1 \otimes k) + i \otimes q$, $p \in \mathrm{span}\{j, k\}$, $q \in \mathrm{span}\{i, j\}$, $a, b \in R$.
- $p \otimes k + a(j \otimes 1) + b(1 \otimes k) + j \otimes q$, $p \in \mathrm{span}\{i, k\}$, $q \in \mathrm{span}\{i, j\}$, $a, b \in R$.

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